Frequency Response

- Before starting, review phasor analysis, Bode plots ...

  Key concept: small-signal models for amplifiers are *linear* and therefore, cosines and sines are solutions of the linear differential equations which arise from R, C, and controlled source (e.g., G_m) networks.

  It is much more efficient to work with *imaginary exponentials* rather than cosine and sine functions; the measured function *v(t)* is considered (by convention) to be the real part of this imaginary exponential

$$v(t) = v\cos(\omega t + \phi) \rightarrow \mathrm{Re}(ve^{(j\omega t + \phi)}) = \mathrm{Re}(ve^{j\phi}e^{j\omega t})$$

where v is the amplitude and ϕ is the phase of the sinusoidal signal *v(t)*.

The ***phasor V*** is defined as the complex number

$$\boxed{V = ve^{j\phi}}$$

Therefore, the measured function is related to the phasor by

$$\boxed{v(t) = \mathrm{Re}(Ve^{j\omega t})}$$

Circuit Analysis with Phasors

- The current through a capacitor is proportional to the derivative of the voltage:

$$i(t) = C\frac{d}{dt}v(t)$$

  We assume that all signals in the circuit are represented by sinusoids.
  Substitution of the phasor expression for voltage leads to:

$$v(t) \rightarrow Ve^{j\omega t} \qquad \ldots \qquad Ie^{j\omega t} = C\frac{d}{dt}(Ve^{j\omega t}) = j\omega CVe^{j\omega t}$$

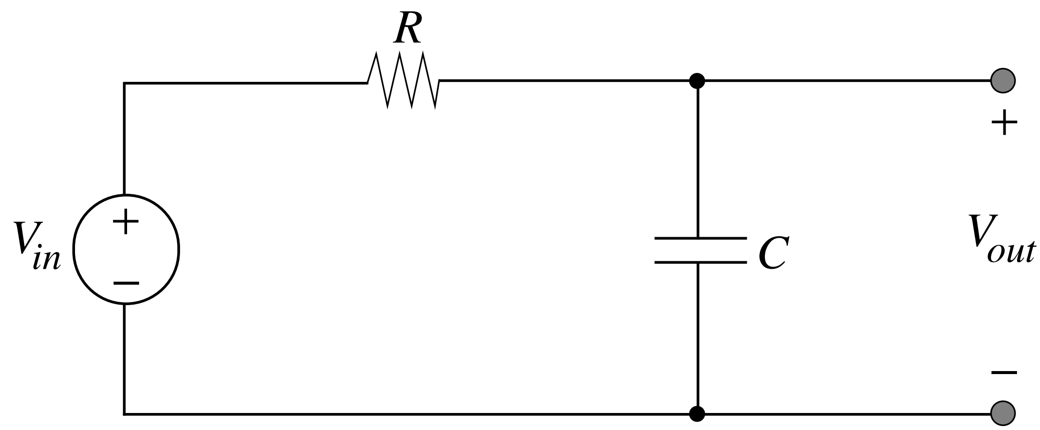
  which implies that the ratio of the phasor voltage to the phasor current through a capacitor (the ***impedance***) is

$$\boxed{Z(j\omega) = \frac{V}{I} = \frac{1}{j\omega C}}$$

- Implication: the phasor current is *linearly proportional* to the phasor voltage, making it possible to solve circuits involving capacitors and inductors as rapidly as resistive networks ... as long as all signals are sinusoidal.

Phasor Analysis of the Low-Pass Filter

■ Voltage divider with impedances --



Replacing the capacitor by its impedance, $1 / (j\omega C)$, we can solve for the ratio of the phasors V_{out} / V_{in}

$$\frac{V_{out}}{V_{in}} = \frac{1/j\omega C}{R + 1/j\omega C}$$

multiplying by $j\omega C / j\omega C$ leads to

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC}$$

Frequency Response of LPF Circuits

- *Bode plots*: magnitude and phase of the phasor ratio: V_{out} / V_{in}

  the range of frequencies is very wide (DC --> 10^8 Hz, for example)
  --> plot frequency axis on log scale

  the range of magnitudes is also very wide (and we care about ratios of 0.001 in some applications):
  --> plot magnitude on log scale

  define magnitude in decibels "dB" by

$$\left|\frac{V_{out}}{V_{in}}\right|_{dB} = 20\log\left|\frac{V_{out}}{V_{in}}\right|$$

  phase is usually expressed in degrees (rather than radians):

$$\angle\frac{V_{out}}{V_{in}} = \operatorname{atan}\left[\frac{Im(V_{out}/V_{in})}{Re(V_{out}/V_{in})}\right]$$

Complex Algrebra Review

* Magnitudes:

$$\left|\frac{Z_1}{Z_2}\right| = \frac{|Z_1|}{|Z_2|} = \frac{\sqrt{X_1^2 + Y_1^2}}{\sqrt{X_2^2 + Y_2^2}}, \text{ where}$$

$$Z_1 = X_1 + jY_1 \qquad Z_2 = X_2 + Y_2$$

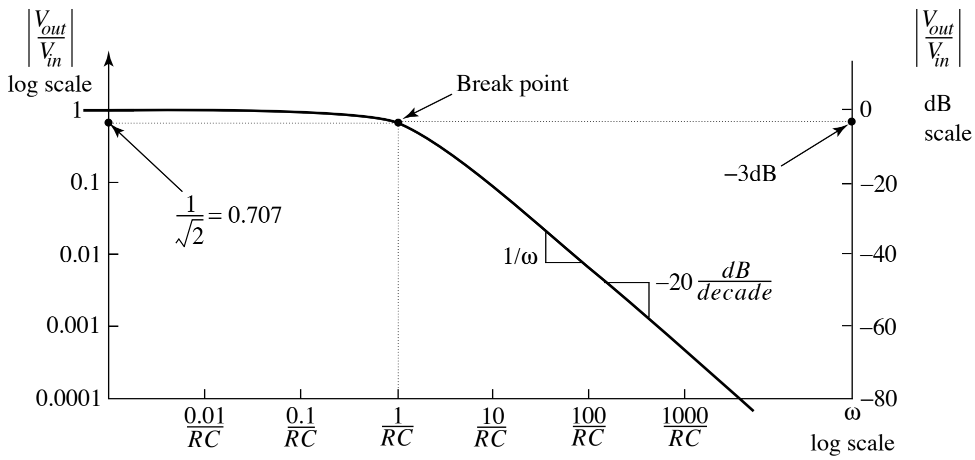
* Phases:

$$\angle\frac{Z_1}{Z_2} = \angle Z_1 - \angle Z_2 = \text{atan}\frac{Y_1}{X_1} - \text{atan}\frac{Y_2}{X_2}$$

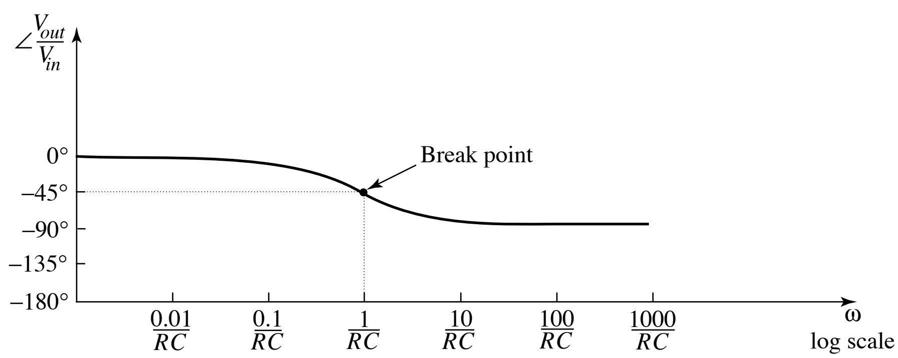
* Examples:

Magnitude and Phase Plots of the Low Pass Filter

- $|V_{out} / V_{in}|$ --> 1 for "low" frequencies; $|V_{out} / V_{in}|$ --> 0 for "high" frequencies



(a)



(b)

The "break point" is when the frequency is equal to $\omega_o = 1 / RC$, at which the ratio of phasors has a magnitude of - 3 dB and the phase is -45^o.

The break frequency defines "low" and "high" frequencies.

Finding the Waveform from the Bode Plot

- Suppose that $v_{in}(t) = 100\text{ mV}\cos(\omega_o t + 0^o)$

  note that the input signal frequency is equal to the break frequency and that the phase is 0^o ... the input signal phase is arbitrary and is generally selected to be 0.

  the output phasor is:

$$V_{out} = V_{in}\left[\frac{1}{1 + j(\omega_o / \omega_o)}\right] = V_{in}\left[\frac{1}{1+j}\right]$$

  magnitude:

$$\left|\frac{V_{out}}{V_{in}}\right|_{dB} = -3\text{dB} \qquad \ldots \qquad |V_{out}| = \frac{|V_{in}|}{\sqrt{2}} = \frac{100\text{mV}}{\sqrt{2}} = 71\text{mV}$$

  phase:

$$\angle\frac{V_{out}}{V_{in}} = \angle 1 - \angle(1+j) = 0 - 45° \qquad \angle V_{out} = -45°$$

$$V_{out} = (71\text{mV})e^{-j45°}$$

  output waveform $v_{out}(t)$ is given by:

$$v_{out}(t) = Re\left(V_{out}e^{j\omega_o t}\right) = Re(71\text{mV}e^{-j45°}e^{j\omega_o t})$$

$$v_{out}(t) = 71\text{ mV}\cos(\omega_o t - 45^o)$$

Bode Plots of General Transfer Functions

- Procedure is to identify standard forms in the transfer functions, apply asymptotic techniques to sketch each form, and then combine the sketches graphically

$$H(j\omega) = \frac{Aj\omega(1 + j\omega\tau_2)(1 + j\omega\tau_{4.})...(1 + j\omega\tau_n)}{(1 + j\omega\tau_1)(1 + j\omega\tau_3)...(1 + j\omega\tau_{n-1})}$$

  where the τ_i are time constants -- $(1/\tau_i)$ are the break frequencies, which are called *poles* when in the demoninator and *zeroes* when in the numerator

- From complex algebra, the factors can be dealt with separately in the magnitude and in the phase and the results added up to find $|H(j\omega)|$ and phase $(H(j\omega))$

  Three types of factors:

  1. poles (binomial factors in the denominator)
  2. zeroes (binomial factors in the numerator)
  3. $j\omega$ in the numerator (or denominator)

Rapid Sketching of Bode Plots

- Poles: - 3 dB and -45^o at break frequency

  0 dB below and -20 dB/decade above

  0^o for low frequencies and -90^o for high frequencies; width of transition is 10 and (1/10) break frequency

- Zeros: +3 dB and +45^o at break frequency

  0 dB below and + 20 dB/decade above

  0^o for low frequencies and +90^o for high frequencies; width of transition is 10 and (1/10) break frequency

- * $j\omega$: +20 dB/decade (0 dB at ω = 1 rad/s) and +90^o contribution to phase

Example: